Outline:

- Repeated roots of the characteristic polynomial
- Method of undetermined coefficients
- Variation of parameters

Last time:

Given a homogeneous linear ODE

$$a_n x^{(n)} + \cdots + a_1 x' + a_0 x = 0,$$

we defined its characteristic polynomial

$$a_n m^n + \cdots + a_1 m + a_0 = 0,$$

and showed that its roots $m_1, \ldots, m_n$ imply solutions to the ODE

$$e^{m_1 t}, \ldots, e^{m_n t}.$$

When the roots are unique, we can find a n-parameter family of solutions

$$x = c_1 e^{m_1 t} + \cdots + c_n e^{m_n t}.$$

This time:

Brief intro to differential operators:

Recall that we can map functions to functions, e.g. $K : C^\infty([0,1]) \to C^\infty([0,1])$ defined by

$$K(x)(t) = x(0) + \int_0^t x(s)\,ds.$$

This notation emphasizes that K is a map.

Let's suppose we have some map $D : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ that happens to be linear.

i.e. $D(x_1 + x_2) = D(x_1) + D(x_2)$

and $D(\alpha x_1) = \alpha D(x_1)$, $\alpha \in \mathbb{R}$.

Often, when unambiguous, we'll drop the parentheses

$$D(x_1 + x_2) = Dx_1 + Dx_2, \quad D(\alpha x_1) = \alpha D x_1.$$

We call D a linear operator mapping functions to functions

$D(x_1+x_2) = Dx_1 + Dx_2$, $D(\alpha x_1) = \alpha D x_1$.

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Let's recall our various notations for derivatives

Leibniz notation: $\frac{dx}{dt}$, $\frac{d^2x}{dt^2}$, $\frac{d^3x}{dt^3}$

Newton notation: $\dot{x}$, $\ddot{x}$, $\dddot{x}$

LaGrange notation: y', y'', y'''

Rewrite the Leibniz notation: (say $D = \frac{d}{dt}$)

$$\frac{dx}{dt} = \frac{d}{dt}x = Dx$$

$$\frac{d^2x}{dt^2} = \frac{d^2}{dt^2}x = \frac{d}{dt}\frac{d}{dt}x = D\cdot D\,x = D^2 x$$

$$\frac{d^3x}{dt^2} = \frac{d^3}{dt^3}x = \frac{d}{dt}\frac{d}{dt}\frac{d}{dt}x = D\cdot D\cdot Dx = D^3 x$$

Turns out that we can think of differentiation in terms of the differential operator.

We can rewrite equations using this: e.g. $\frac{d^2}{dt^2}x + \frac{d}{dt}x = 0$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{d}{dt}+1\right)x = 0$$

Later, we will see how to use this to solve ODEs.

Non-unique roots:

Suppose the roots are not unique. Let's consider a root with multiplicity n. Given

$a_n \cdot \frac{d^n}{dt^n}x + \dots + a_1\frac{d}{dt}x + a_0 = 0$ with char eq. $a_n m^n + \dots + a_1 m + a_0 = 0$,

suppose we can rewrite it (in operator notation)

$a_n\left(\frac{d}{dt} - r\right)^n x = 0$ with char eq. $a_n(m-r)^n = 0$.

This tells us that ce^{rt} is a solution, for any constant c, but this is only a 1-parameter family of solutions.

Can we find an n-parameter family?

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Can we find an n-parameter family?

Let's guess that $u(t)e^{rt}$ is a solution. (replacing the constant with a function)

$$a_n\left(\frac{d}{dt}-r\right)^n(ue^{rt})=0$$

$$\Rightarrow\left(\frac{d}{dt}-r\right)^n(ue^{rt})=0$$

Note that $\left(\frac{d}{dt}-r\right)(ue^{rt})=\frac{d}{dt}(ue^{rt})-rue^{rt}=rue^{rt}+e^{rt}\frac{d}{dt}u-rue^{rt}=e^{rt}\frac{d}{dt}u$

$$\Rightarrow\left(\frac{d}{dt}-r\right)^n(ue^{rt})=e^{rt}\frac{d^n}{dt^n}u=0.$$

$$\Rightarrow\frac{d^n}{dt^n}u=0.$$

Let's integrate n times:

$$u=c_0+c_1t+\cdots+c_{n-1}t^{n-1}$$

Thus $x_c=(c_0+c_1t+c_2t^2+\cdots+c_{n-1}t^{n-1})e^{rt}$ is a solution to the ODE

for any constants $c_1,\dots,c_{n-1}$, so we have an n-parameter family.

Alternately, we can think of us having n independent solutions

$$e^{rt},\ te^{rt},\ t^2e^{rt},\dots,\ t^{n-1}e^{rt}$$

if we have a repeated root r of multiplicity n.

We proved it for an ODE with a single root, but a similar argument can be made for multiple roots with >1 multiplicity.

Ex. $$\frac{d}{dt^2}\left(\frac{d}{dt}-a\right)^3\left(\frac{d}{dt}+b\right)^4\left(\frac{d}{dt}+c\right)=0$$

Char eq: $m^2(m-a)^3(m+b)^4(m+c)=0$

$m=0$, multiplicity 2

a, multiplicity 3

$-b$, multiplicity 4

$$-c\ ,\quad \text{multiplicity } 1$$

$$\Rightarrow \overset{1}{\underset{}{e^{0t}}},\ \overset{t}{t e^{0t}} \quad \text{are solutions corresponding to } m=0$$

$$e^{at},\ te^{at},\ t^2e^{at} \quad \text{" " "} \quad m=a$$

$$e^{-bt},\ te^{-bt},\ t^2e^{-bt},\ t^3e^{-bt} \quad m=-b$$

$$e^{-ct} \quad m=-c$$

So
$$x_c = c_1 + c_2 t + c_3 e^{at} + c_4 te^{at} + c_5 t^2 e^{at} + c_6 e^{-bt} + c_7 te^{-bt} + c_8 t^2 e^{-bt} + c_9 t^3 e^{-bt} + c_{10} e^{-ct}$$

Ex. $y'''' - 3y'' + 2y' = 0$

char eq. $m^4 - 3m^2 + 2m = 0$

$$m(m^3 - 3m + 2) = 0$$

$$m(m^3 - m^2 + m^2 - 3m + 2) = 0$$

$$m[m^2(m-1) + (m-1)(m-2)] = 0$$

$$m(m-1)(m^2 + m - 2) = 0$$

$$m(m-1)(m-1)(m+2) = 0$$

$$m(m-1)^2(m+2) = 0$$

$$m = 0,\ 1\ (\text{mul } 2),\ -2$$

$$\Rightarrow e^{0t} = 1,\ e^t,\ te^t,\ e^{-2t} \quad \text{are solutions}$$

$$\Rightarrow y_c = c_1 + c_2 e^t + c_3 te^t + c_4 e^{-2t}$$

Inhomogeneous Equations

Recall $a_n x^{(n)} + \cdots + a_1 x' + a_0 x = Q(t)$, where $a_n \neq 0$ and $Q(t) \neq 0$

has an n-parameter family of solutions

$$x = x_c + x_p\ ,$$

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where x_c is an n-parameter family of solutions to the homogeneous eqn and x_p is any particular solution.

How can we find x_p?

Method of undetermined coefficients

Only works if $Q(t)$ has finitely many ind. derivatives $Q, Q', Q'', \ldots, Q^{(k)}$

We assume that x_p is a linear combination of $Q, Q', \ldots, Q^{(k)}$, and solve.

e.g. $\ddot{x} + \omega_0^2 x = \sin t$. Recall $x_c = c_1 \sin(\omega_0 t) + c_2 \cos(\omega_0 t)$

$$Q(t) = \sin t$$
$$Q'(t) = \cos t$$
$$Q''(t) = -\sin t \quad \leftarrow \text{not linearly ind. of the first 2.}$$

So
$$x_p = A \sin t + B \cos t$$
$$\dot{x}_p = A \cos t - B \sin t$$
$$\ddot{x}_p = -A \sin t - B \cos t$$

$$\Rightarrow (-A \sin t - B \cos t) + \omega_0^2 (A \sin t + B \cos t) = \sin t$$

Match together coefficients of $\sin t$ and of $\cos t$

$$-A + \omega_0^2 A = 1\,, \qquad -B + B\omega_0^2 = 0$$

$$A = \frac{1}{\omega_0^2 - 1} \qquad\qquad B = 0$$

$$\Rightarrow x_p = \frac{1}{\omega_0^2 - 1} \sin t$$

$$\Rightarrow x = \frac{1}{\omega_0^2 - 1} \sin t + c_1 \sin(\omega_0 t) + c_2 \cos(\omega_0 t)$$

Why does this work? Because the LHS has to have terms corresponding to the RHS. But new linearly ind. terms can be generated from taking derivatives, so we need to be sure we can cancel all of them out which is why to the RHS. But new linearly ind. terms can be generated from taking derivatives, so we need to be sure we can cancel all of them out, which is why the method of undetermined coefficients only works when $Q(x)$ and its derivatives can be generated from a finite set of basis functions.

Case 1: No term in the RHS $Q(t)$ is the same as a term in the complementary solution $x_c(t)$. Then $x_p(t)$ is a linear combination of $Q, Q', Q'', \dots$

Ex. $y'' + 4y' + 4y = 4x^2 + 6e^x$

Char. eq.: $m^2 + 4m + 4 = 0 \Rightarrow m = -2$ (double root)

So e^{-2x}, xe^{-2x} are solutions to the homogeneous eq.

$\Rightarrow y_c = c_1 e^{-2x} + c_2 x e^{-2x}$

Guess $y_p = Ax^2 + Bx + C + De^x$

$y_p' = 2Ax + B + De^x$

$y_p'' = 2A + De^x$

$$(2A + De^x) + 4(2Ax + B + De^x) + 4(Ax^2 + Bx + C + De^x) = 4x^2 + 6e^x$$

$$x^2(4A) + x(8A + 4B) + (2A + 4B + 4C) + e^x(D + 4D + 4D) = 4x^2 + 6e^x$$

$4A = 4 \Rightarrow A = 1$

$8A + 4B = 0 \Rightarrow B = -2$

$2A + 4B + 4C = 0 \Rightarrow C = \frac{3}{2}$

$9D = 6 \Rightarrow D = \frac{2}{3}$

$\Rightarrow y_p = x^2 - 2x + \frac{3}{2} + \frac{2}{3}e^x$

So the general solution is $y_c + y_p$

$\Rightarrow y = (c_1 + c_2 x)e^{-2x} + x^2 - 2x + \frac{3}{2} + \frac{2}{3}e^x$

Case 2: A term in the RHS is $t^k u(t)$, $k \in \{0, 1, 2, \dots\}$, and $u(t)$ is a term in the complementary solution x_c (up to constant multiplication). Then you have to add the derivatives of $t^{k+1}u(t)$ to the basis of x_p.

Then you have to add the derivatives of $t^{k+1}u(t)$ to the basis of x_p.

Ex. $\ddot{x}(t) - 3\dot{x}(t) + 2x(t) = 2t^2 + 3e^{2t}$

Char. eq. $m^2 - 3m + 2 = 0$

$m = 1, 2$ are roots

So $x_c = c_1 e^t + c_2 e^{2t}$.

Suppose we tried using the method from case 1.

Then $x_p = At^2 + Bt + C + De^{2t}$

This term is in the complementary solution, so it disappears when we plug it in, and we cannot cancel out $3e^{2t}$ on the RHS.

Thus, we guess instead that

$$x_p = At^2 + Bt + C + Dte^{2t} \cancel{+ Ee^{2t}}$$

this term still disappears, so we don't need it.

$$\dot{x}_p = 2At + B + D(2te^{2t} + e^{2t}) = 2At + B + 2Dte^{2t} + De^{2t}$$

$$\ddot{x}_p = 2A + 2D(2te^{2t} + e^{2t}) + 2De^{2t} = 2A + 4Dte^{2t} + 4De^{2t}$$

$$
\begin{aligned}
& 2A + 4Dte^{2t} + 4De^{2t} \\
& -6At - 3B - 6Dte^{2t} - 3De^{2t} \\
& +2At^2 + 2Bt + 2C + 2Dte^{2t} \\
= \; & 2t^2 + 3e^{2t}
\end{aligned}
$$

$$\Rightarrow \left.\begin{aligned} A &= 1 \\ B &= 3 \\ C &= \tfrac{7}{2} \\ D &= 3 \end{aligned}\right\} \quad x_p = t^2 - 3t + \frac{7}{2} + 3te^{2t}$$

The general solution is

$$x = c_1 e^t + c_2 e^{2t} + t^2 - 3t + \frac{7}{2} + 3te^{2t}$$

$$x = c_1 e^{t} + c_2 e^{2t} + t^2 - 3t + \frac{7}{2} + 3te^{2t}$$

Case 3: A term in the RHS is $t^k u(t)$, where $u(t)$ is a term in the complementary solution corresponding to a multiple root with multiplicity r of the characteristic equation. Then we have to add the derivatives of $t^{k+r} u(t)$ to the basis for x_p.

Ex. $y'' + 4y' + 4y = 3xe^{-2x}$

Char. eq. $m^2 + 4m + 4 = 0$

$m = -2$ (double root)

$$y_c = c_1 e^{-2x} + c_2 x e^{-2x}$$

part of y_c

$$y_p = Ax^3 e^{-2x} + Bx^2 e^{-2x} + \cancel{Cxe^{-2x} + De^{-2x}}$$

$$y_p' = -2Ax^3 e^{-2x} + 3Ax^2 e^{-2x} - 2Bx^2 e^{-2x} + 2Bxe^{-2x}$$

$$= -2Ax^3 e^{-2x} + (3A - 2B)x^2 e^{-2x} + 2Bxe^{-2x}$$

$$y_p'' = 4Ax^3 e^{-2x} - 6Ax^2 e^{-2x} - 2(3A - 2B)x^2 e^{-2x} + 2(3A-2B)xe^{-2x} - 4Bxe^{-2x} + 2Be^{-2x}$$

$$= 4Ax^3 e^{-2x} + (-12A + 4B)x^2 e^{-2x} + (6A - 8B)xe^{-2x} + 2Be^{-2x}$$

Then

$$\begin{aligned} & 4Ax^3 e^{-2x} + (-12A + 4B)x^2 e^{-2x} + (6A - 8B)xe^{-2x} + 2Be^{-2x} \\ & -8Ax^3 e^{-2x} + (12A - 8B)x^2 e^{-2x} + 8Bxe^{-2x} \\ & +4Ax^3 e^{-2x} + 4Bx^2 e^{-2x} \\ = \; & 3xe^{-2x} \end{aligned}$$

$$\Rightarrow A = \frac{1}{2}, \quad B = 0$$

$$\Rightarrow y_p = \frac{1}{2} x^3 e^{-2x}$$

$$\Rightarrow y = \frac{1}{2} x^3 e^{-2x} + c_1 e^{-2x} + c_2 x e^{-2x}.$$

Note that we have been taking advantage of linearity and the principle of superposition to solve for the RHS. We can use this to rewrite real problems as easier to solve complex ones.

Given $a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = Q(x)$, $a_n \neq 0$, $a_i \in \mathbb{R}$

if y_p is a particular solution, then

$Re(y_p)$ is a particular soln to $a_n y^{(n)} + \cdots + a_1 y' + a_0 y = Re(Q(x))$

$Im(y_p)$ " " " " " " " " $= Im(Q(x))$.

Ex. Find a particular solution to

$$y'' - 3y' + 2y = \sin x.$$

- We can solve this directly using the method of undetermined coefficients:

$$\left.\begin{aligned} y_p &= A\sin x + B\cos x \\ y_p' &= A\cos x - B\sin x \\ y_p'' &= -A\sin x - B\cos x \end{aligned}\right\} \quad \begin{aligned} &-A\sin x - B\cos x \\ &+3B\sin x - 3A\cos x \\ &+2A\sin x + 2B\cos x \\ &= \sin x \end{aligned}$$

$$\Rightarrow A + 3B = 1, \quad B - 3A = 0$$

$$\Rightarrow A = \frac{1}{10}, \quad B = \frac{3}{10}.$$

$$y_p = \frac{1}{10}\sin x + \frac{3}{10}\cos x$$

- Or, we can solve by using complex variables.

Let's solve $y'' - 3y' + 2y = e^{ix}$.

(The coefficient is complex too)

$$\left.\begin{aligned} y_p &= (A+Bi)e^{ix} \\ y_p' &= (-B+Ai)e^{ix} \\ y_p'' &= (-A-Bi)e^{ix} \end{aligned}\right\} \quad \left.\begin{aligned} &(2A+2Bi)e^{ix} \\ &(-3B-3Ai)e^{ix} \\ &(-A-Bi)e^{ix} \\ &= e^{ix} \end{aligned}\right\} \quad \left.\begin{aligned} A - 3B &= 1 \\ B - 3A &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} A &= \tfrac{1}{10} \\ B &= \tfrac{3}{10} \end{aligned}$$

$$y_p = \left(\frac{1}{10} + \frac{3}{10}i\right)e^{ix} = \left(\frac{1}{10} + \frac{3}{10}i\right)(\cos x + i\sin x)$$

$$Im(y_p) = \frac{1}{10}\sin x + \frac{3}{10}\cos x.$$

Sometimes, it'll be easier to solve using complex variables and then taking the real or imaginary parts.

Method of Variation of Parameters

Given $f_n(x)y^{(n)} + f_{n-1}(x)y^{(n-1)} + \cdots + f_1(x)y' + f_0(x)y = Q(x)$,

Method of Variation of Parameters

Given $f_n(x)y^{(n)} + f_{n-1}(x)y^{(n-1)} + \cdots + f_1(x)y' + f_0(x)y = Q(x)$, where $f_i(x)$ is continuous and $f_n(x) \neq 0$ for any x on the interval, if we can find n linearly independent solutions $y_1, \ldots, y_n$ to the homog. eqn.

$$f_n(x)y^{(n)} + f_{n-1}(x)y^{(n-1)} + \cdots + f_1(x)y' + f_0(x)y = 0,$$

then we can find a particular solution y_p of the form

$y_p(x) = u_1(x)y_1(x) + \cdots + u_n(x)y_n(x)$, where $u_i(x)$ are unknown functions.

We can then solve for $u_i(x)$ by plugging y_p back into the original ODE, which will give the system of equations

$$\begin{aligned}
u_1'y_1 + \cdots + u_n'y_n &= 0 \\
u_1'y_1' + \cdots + u_n'y_n' &= 0 \\
&\vdots \\
u_1'y_1^{(n-2)} + \cdots + u_n'y_n^{(n-2)} &= 0 \\
u_1'y_1^{(n-1)} + \cdots + u_n'y_n^{(n-1)} &= \frac{Q(x)}{f_n(x)}.
\end{aligned}$$

We can then solve for each u_i', and then integrate to get u_i.

Note: This method works for non-constant coefficients and when $Q(x)$ has infinitely many linearly ind. derivatives, but it can be harder to work with than undetermined coefficients.

Ex. $y'' - 3y' + 2y = \sin e^{-x}$

What if we try method of undetermined coefficients?

$$\left.\begin{aligned}
Q(x) &= \sin e^{-x} \\
Q'(x) &= -e^{-x}\cos e^{-x} \\
Q''(x) &= -e^{-2x}\sin e^{-x} + \cdots \\
Q'''(x) &= e^{-3x}\cos e^{-x} + \cdots \\
&\vdots
\end{aligned}\right\}$$ linearly ind., so we can't use method of undetermined coefficients

Let's use variation of parameters insted.

First need 2 ind. solutions to the homog. eqn.

Let's use variation of parameters insted.

First need 2 ind. solutions to the homogeneous eqn

$$y''-3y'+2y=0.$$

Char eq. $m^2-3m+2=0 \Rightarrow m=1,2,$

so e^x, e^{2x} are linearly ind. soln. to the homogeneous eqn

Then $y_p = u_1 e^x + u_2 e^{2x}$, and

$$\begin{cases} u_1' e^x + u_2' e^{2x} = 0 \\ u_1' e^x + 2u_2' e^{2x} = \sin e^{-x} \end{cases}$$

$\Rightarrow u_1' e^x = -u_2' e^{2x}$, $u_1' = -u_2' e^x$

$\Rightarrow u_2' e^{2x} = \sin e^{-x}$

$u_2' = e^{-2x} \sin e^{-x}$, $u_1' = -e^{-x} \sin e^{-x}$

Integrate letting $u = e^{-x}$, $du = -e^{-x} dx$, $dx = -e^x du = -\frac{du}{u}$

$$\frac{du_2}{dx} = e^{-2x} \sin e^{-x} \qquad \frac{du_1}{dx} = -e^{-x} \sin e^{-x}$$

$$du_2 = e^{-2x} \sin e^{-x}\, dx \qquad du_1 = -e^{-x} \sin e^{-x}\, dx$$

$$du_2 = -u \sin u\, du \qquad du_1 = \sin u\, du$$

Integration by parts

$$u_2 = -\sin u + u \cos u \qquad u_1 = -\cos u$$

$$u_2 = -\sin e^{-x} + e^{-x} \cos e^{-x} \qquad u_2 = -\cos e^{-x}$$

$$\Rightarrow y_p = (-\cos e^{-x}) e^x + (-\sin e^{-x} + e^{-x} \cos e^{-x}) e^{2x}$$
$$= -e^{2x} \sin e^{-x}.$$

Thus, $y = c_1 e^x + c_2 e^{2x} - e^{2x} \sin e^{-x}$. $\square$